Statistical Signal Analysis Using Wavelets

Year 1 Report

ONR Contract Number: N00014-93-C-0106

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January 17, 1995

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Final Report

Executive Summary

This report discusses the research on "Statistical Signal Analysis Using Wavelets" performed by the Statistical Sciences Division of MathSoft, Inc. for contract N00014-93-C-0106 with the Office of Naval Research. The overall goals of the research are:

- Application of wavelets and related transforms to data analysis.
- Using wavelets as the basis for statistical problems, such as signal extraction, spectral density estimation, isotonic regression, and classification.
- Research into new transforms and algorithms tailored to meet the needs of data analysis and statistical estimation.

The first year of research focused on six specific areas:

1. Applications of wavelets to data of interest to the Navy;
2. Noise removal using wavelets, wavelet packets, and cosine packets;
3. Investigation into new wavelet transforms which are outlier resistant and edge preserving;
4. Development of a framework and tools for the "wavelet approach" towards analysis of signals, images, and other data;
5. Exploration of the use of wavelets as a dimension reduction tool for statistical analysis of very large data sets;
6. Development of algorithms for wavelet analysis.

Future research will extend the results obtained in these areas. In addition, research in the following areas will be pursued, time permitting: analysis of wavelet coefficients for fractional Brownian motion, and simulation, bootstrapping, and modeling of non-Gaussian processes.

Contents

1 INTRODUCTION

In the past few years, wavel ts have evolved from an interesting mathematical discovery to a valuable technique with a wide variety of applications. Wavelet research has synthesized a number of related ideas into a coherent set of tools and methodology for analysis of signals, images and other technical data. Research is leading to new wavelet-like methods, such as wavelet packets or outlier-resistant wavelets. The development of software toolkits are bringing these methods to the hands of scientists and engineers, including those without wavelet expertise.

Our research for ONR contract N00014-93-C-0106 has focused on:

- Application of wavelets and related transforms to data analysis.
- Using wavelets as the basis for statistical problems, such as signal extraction, spectral density estimation, isotonic regression, and classification.
- Research into new transforms and algorithms tailored to meet the needs of data analysis and statistical estimation.

Our research in each of these areas is guided by data analysis of time series of interest to the Navy.

In section 2, we summarize the main results from the first year of research. Section 3 gives a list of all papers and talks produced during the year. Future research plans are discussed in section 4. We also include an appendix with plots giving examples and illustrations.

2 SUMMARY OF RESULTS

We are pursuing research in six related areas:

1. Applications of wavelets to data of interest to the Navy;
2. Noise removal using wavelets, wavelet packets, and cosine packets;
3. Investigation into new wavelet transforms which are outlier resistant and edge preserving;
4. Development of a framework and tools for the "wavelet approach" towards analysis of signals, images, and other data;
5. Exploration of the use of wavelets as a dimension reduction tool for statistical analysis of very large data sets;
6. Development of algorithms for wavelet analysis.

These are discussed in more detail below.

2.1 DATA ANALYSIS WITH WAVELETS

Applications of interest to the Navy include:

- Robust wavelet de-noising of radar glint noise.
- Time-frequency analysis of underwater acoustic signals.
- Wavelet compression applied to a number of signals and images.
- Fast classification of transients in low frequency sinusoidal data.

Some of these applications are discussed in more detail in the following sections.

We are currently writing up a series of reports on applications of wavelets to data analysis problems [BG95c, BG95d, BG95e]. Our analysis of Navy data is also closely related to our work in developing the "wavelet approach" to data analysis: see section 2.4.

2.2 NOISE REMOVAL WITH WAVELETS

In the past three years, wavelet de-noising research has received intense activity since the initial development by Donoho and Johstone. This research includes methods for the following topics.

Signal recovery: The radar glint noise example is an example of signal recovery from noisy data. The underlying model for our data y_i is $y_i = f_i + \epsilon_i$ where f_i is the unknown signal or image and ϵ_i is noise. Wavelets provide a way to obtain an estimate $\hat{f}_i$ while making a minimum of assumptions about the nature of f_i and ϵ_i. Other applications have been explored by [Bri94, CM92, DJ92a, DJ92b, MH92, Don93, MZ93, cA94, HK94, IK93, PBBA94, RBL94, SS94, TH94, Tew94].

Inverse Problems (Parameter Estimation): Many interesting problems having to do with noisy data involve *indirect* measurements $y_i = (Kf)(t_i)+\epsilon_i$ where we want to estimate f (so called inverse problems). Examples of the transform K include the Fourier transformation (magnetic resonance imaging), the Laplace transformation (fluorescence spectroscopy), the Radon transformation (tomography problem) and various deconvolution problems (gravity anomalies, infrared spectroscopy, extragalactic astronomy). See [Don93, Wic94, BFCLB94, MW94] for applications of wavelets to inverse problems.

Signal Detection and Classification: Closely related to the problem of signal recovery is signal detection and classification. The aim is to identify a particular signal from background noise and other signals. Wavelet based signal detection has been explored by [FM92, LKW92, KDP92, YRKS92, Car94, DMWJ94, LAJ94].

Image and Image Array Reconstruction and Enhancement: Wavelet de-noising methodology can be extended to two and higher dimensional problems in a straightforward manner. Some applications in which wavelets have been used to clean noisy images are given in [DJL92, SFAH92, MH92, Don93, WRM+94].

Density estimation: Wavelets can be used to recover probability densities [JKP92] and spectral densities [Mou93, Gao93b] from noisy data (see section 2.2.5).

In year two research, we have studied

1. Wavelet de-noising using non-decimated transforms.
2. Variance estimation for wavelet de-noising.
3. Selecting the threshold using cross-validation.
4. Applications of wavelets to isotonic regression.
5. Spectral density estimation using wavelets.
6. Diagnostics for assessing wavelet de-noising.

This are discussed in more detail below.

2.2.1 Wavelet De-Noising using Non-Decimated Transforms

The non-decimated discrete wavelet transform is a non-orthogonal variant of the classical DWT. With the non-decimated DWT, starting with n sample signal values you end up with $(J+1) \times n$ coefficients. Unlike the classical DWT, which has fewer coefficients at coarser scales, each scale for the non-decimated DWT has n coefficients. The non-decimated DWT is *over-sampled* at coarse scales. This over-sampling can enhance the visual displays and lead to advantages in certain problems, including better spatial resolution, less loss of information, and translation invariance with respect to the signal. Mallat and co-workers used the non-decimated DWT for detecting singularities and de-noising signals [MH92, MZ92].

Instead of using an orthogonal discrete wavelet transform, the Donoho and Johnstone WaveShrink algorithm can be applied to the non-decimated wavelet transform. Figure 1 illustrates this using a noisy sinusoid with a discontinuity. The WaveShrink estimate based on the orthogonal `haar` wavelet transform is very blocky, corresponding to the discontinuous nature of the `haar` wavelet. By contrast, the WaveShrink estimate based on based on the non-decimated `haar` wavelet estimate is smooth, due to the "spatial averaging" in the reconstruction. There is, however, a cost to smoothness: the non-decimated estimate blurs the discontinuities at the two jumps.

Preliminary results indicate that de-noising with non-decimated wavelets offers significant improvements over the original WaveShrink algorithm. We plan to continue our research into de-noising with non-decimated wavelets, through both an empirical study and the development of software for use by Navy researchers at the China Lake facility.

2.2.2 Variance Estimation for Wavelet De-Noising

Based on ideas proposed by David Brillinger [Bri94], we have developed a general method for estimating the variance of WaveShrink estimate. Let $\hat{f}_t$ be the WaveShrink estimate. Since the wavelet transform is linear and the wavelet coefficients $w_{j,k}$ are approximately uncorrelated, asymptotic theory implies that

$$\mathrm{Var}(\hat{f}_t) \approx \sum_{j,k} \mathrm{Var}(\hat{w}_{j,k}) \tag{1}$$

where

$$\hat{w}_{j,k} = \delta_\lambda(w_{j,k}) w_{j,k} \|\psi_{j,k}(t)\| \tag{2}$$

In figure 2, we apply this formula to compute a confidence interval for the flow of the Nile River. The top plot is the annual Nile River flow at Aswan from 1875 to 1970, the middle plot is the WaveShrink using Haar wavelet with an approximated 95% confidence band and the bottom plot is the WaveShrink using *bs1.3* wavelet with an approximated 95% confidence band. The confidence interval clearly shows a significant drop around the time of the building of the Aswan dam in 1899-1902.

To make (1) more generally useful, we plan to investigate several extensions. These include the development of a fast algorithm to compute $\|\psi_{j,k}(t)\|$ and use of the bootstrap to get a more realistic estimate of $\mathrm{Var}(\hat{w}_{j,k})$.

2.2.3 Cross Validation Selection of the Threshold

Initial simulations have shown that the cross validation technique is promising in selecting thresholds in the WaveShrink algorithm. The basic idea is:

[1] for each set of thresholds, apply WaveShrink to part of the data and compute the mean-square error between the WaveShrink estimate and the other part of the data;

[2] find the set of thresholds which minimizes the mean-square errors.

In figure 3, we apply cross validation to select the threshold for WaveShrink applied to the synthetic "doppler" signal corrupted by Gaussian noise. The cross-validation threshold is compared with the "universal" threshold, "minimax" threshold and the "optimal" threshold (the threshold minimizes the mean-square error

(MSE) between WaveShrink estimate and the true signal, this is only possible for simulation). Cross validation consistently leads to lower MSE than the "universal" and "minimax" thresholds for this example.

In future research, we will to explore using non-linear optimization techniques to estimate multiple thresholds with cross validation. We will also explore other cross validation methods, such as using decimation by powers of 2^j instead of just decimation by 2 (initial investigations have not shown this to be crucial, but we suspect this may not be the case for "real world" problems). We will also develop a theoretical justification for cross-validation in the WaveSrhink context.

2.2.4 Wavelets and Isotonic Regression

Consider the isotonic regression model:

$$y_i = f(t_i) + z_i$$

where f is a decreasing function and $\{z_i\}$ are assumed to be a stationary Gaussian process with mean zero and variance σ^2. We propose a simple thresholding procedure based on the fact that the wavelet coefficients for f, under Haar wavelet, are non-negative. We show that our estimator is competitive with the Grenander estimator both theoretically and numerically (in the sense of mean-square-error). Figure 4 displays a synthetic decreasing curve (top left), the same curve plus Gaussian white noise (top right), the Grenander estimate (bottom left) and the wavelet estimate (bottom right). The wavelet estimate has lower mean square error and does a better job of preserving the jumps.

Details of the wavelet estimation procedure are given in [Gao95b]: see section 3.

A limitation of the current procedure is the restriction to the use of the Haar wavelet, which is the only wavelet which preserves the monotonicity. We will investigate the existence of a new class of smooth wavelets which preserve monotonicity.

2.2.5 Wavelet Based Spectral Density Estimation

A technique for spectral density estimation based on wavelet decomposition of the periodogram and reconstruction of the spectrum was developed by Hong-Ye Gao [Gao93b, Gao93a]. This technique is especially valuable for processes with non-smooth features in the spectrum, such as sharp peaks.

A simulated autoregressive series is displayed in figure 5. This series has sharp peaks in its spectral density function. Figure 6 compares wavelet shrinkage estimation applied to this series to a more traditional non-parametric spectral estimator based on a kernel smoother. The traditional nonparametric estimator based on a triangular spectral smoothing window tends to oversmooth the peaks. Using shorter span smoothing windows would better preserve the peaks, but would result an unnecessarily rough estimate elsewhere. The WaveShrink estimator is equivalent to

using a variable bandwidth smoother. It preserves the peaks while producing a smooth estimate elsewhere.

A paper describing Dr. Gao's work has been accepted by the Journal of Time Series Analysis pending revisions [Gao95a] (see section 3). These revisions are being done through the support of this contract.

2.2.6 Diagnostics for Assessing Wavelet De-Noising

The theory for WaveShrink is based on the Gaussian white noise model. When this model is inappropriate – e.g., when the data has outliers – then the "traditional" WaveShrink estimators may be inappropriate.

To help assess the WaveShrink fit, we have explored the use of simple diagnostics plots. Figure 7 displays a synthetic bumps signal, the bumps signal corrupted by Gaussian noise, and the WaveShrink estimate of the bumps signal. How should we assess the WaveShrink estimate? Figure 8 gives four "views" of the WaveShrink estimate for the bumps signal: (1) The decomposition of the data into signal plus noise, (2) Boxplots of the DWT coefficients for the original data with the WaveShrink thresholds superimposed as horizontal lines, (3) The DWT of the signal, and (4) A barplot showing the decomposition of the energy of the data into the energy attributable to signal and residual energy.

We can also use plots to examine the residuals from the figure. Figure 9 displays a series of views for the residuals from the WaveShrink estimate of the bumps signal. The diagnostic plots indicate that the peaks are oversmoothed. As a result, the distribution of the residual component is skewed toward high values. The oversmoothing also leads to significant autocorrelation in the residuals.

We will continue to explore the use of visual diagnostics to help guide the user when to use WaveShrink, and how to improve the WaveShrink fit.

2.3 ROBUST AND NONLINEAR WAVELETS

The aim of this research is to investigate wavelet methods which are robust towards outliers. This line of research was motivated by problems encountered in application of wavelets to Navy data sets. It promises to be of considerable practical importance, and represents an exciting area of innovative research.

Some outlier resistant and edge preserving wavelets we developed include:

1. *Wavelets with robust smoother/cleaner.* The usual wavelet transform is combined with a robust smoother/cleaner to remove outliers.

2. *Wavelet based minimum entropy segmentation.* The wavelet transform is used to segment a noisy signal by successively identifying and removing edges and other discontinuities.

These are discussed in more detail below.

Research in this area is still continuing: see sections 3.6 and 4.

2.3.1 Wavelets with Robust Smoother/Cleaner

The presence of outliers in data causes problems in traditional time series analysis techniques. Outliers can seriously distort the autocorrelation function, partial autocorrelation function, spectral density function, model identification, and parameter estimates for models. Outliers can also cause problems with methods based on the wavelet decomposition. Wavelets are a linear transformation of the data, and hence, outliers have unbounded influence on the wavelet coefficients.

The goal of robust smoother/cleaner wavelets is to produce a fast wavelet decomposition which is robust towards outliers. Smoother-cleaner wavelets behave like the classical L_2 wavelet transform for Gaussian signals, but prevent outliers and outlier patches from leaking into the wavelet coefficients at coarse levels (like L_1 wavelets). However, in contrast to the L_1 wavelets, algorithm is very fast with computational complexity $O(n)$.

The basic idea of robust smoother/cleaner wavelets is simple: the smooth coefficients are preprocessed with a fast and robust smoother/cleaner. The procedure is illustrated in figure 10. As usual, we start with a set of wavelet coefficients $S(0)$. Then, for each multiresolution level, the signal is decomposed into three components:

1. A set of robust residuals $R(\ell-1)$, given by

$$R(\ell-1) = \delta_\lambda \left(S(\ell-1) - \hat{S}(\ell-1)\right)$$

   where δ_λ is a thresholder function and $\hat{S}(\ell-1)$ is a robust smooth of $S(\ell-1)$ (e.g., running medians of 5). The threshold λ is chosen so that most of the robust residuals are zero.

2. The smooth wavelet coefficients $S(\ell)$ obtained by applying the usual low-pass/decimation wavelet filter H to the cleaned smooth coefficients $U(\ell-1) = S(\ell-1) - R(\ell-1)$.

3. The detail wavelet coefficients $D(\ell)$ obtained by applying the usual high-pass/decimation wavelet filter G to $U(\ell-1)$.

The smoother/cleaner wavelet decomposition can be used for de-noising signals. As an example, we apply it to the radar glint noise signal. The original noisy signal, which is the angle of the target in degrees, is displayed in Figure 11(a) The true signal is a low-frequency oscillation about $0°$. The signal contains a number of glint spikes, causing the apparent signal to be different from the true signal by as much as $150°$.

Figure 11(b) compares denoising with linear shrinkage of wavelets (dashed line) to denoising with WaveShrink combined with robust smoother-cleaner wavelets (solid line). The linear shrinkage is based on annihilating all detail coefficients of the classical wavelet transform at levels $\ell = 1, 2, 3, 4$. While linear shrinkage estimate is smooth, it is still somewhat sensitive to the glint spikes. By contrast, the clean and repeat procedure is quite resistant to the glint spikes but effectively tracks the sudden changes in the series.

This work led to publication of two papers [BDGM94a, BDGM94b]: see section 3. Refer to these papers for details about the smoother-cleaner wavelets.

2.3.2 Segmented Wavelet Bases

Another way to generalize wavelet bases is through segmentation. Segmented wavelet bases are obtained by piecing together different wavelet bases over time and space. Segmented bases are particularly well suited for adaptation to discontinuities (edges, cusps, etc.) and more general change points (e.g., changes in the stochastic properties of the signal). The segmentation is done by optimizing some criterion (e.g., entropy) in a similar manner as in the search for a "best basis".

In work supported by this contract, the paper "On Minimum Entropy Segmentation" will appear in a future volume of *Wavelet Analysis and Its Applications*, edited by Charles Chui [Don95] (see section 3).

2.4 A FRAMEWORK FOR THE "WAVELET APPROACH"

In the late 1960's and early 1970's, John Tukey developed and popularized a philosophy for analyzing data called "exploratory data analysis" [Tuk75]. Since that time, this philosophy has expanded and matured into a complete collection of tools and techniques for understanding and extracting information from data. Our belief is that wavelet analysis will follow a similar evolution. What started as an interesting mathematical breakthrough has evolved into a coherent new way of analyzing signals, images, and other data.

Many of the ideas behind wavelets are drawn from other domains, such as subband filtering, approximation theory, signal processing, and image processing. What is significant about wavelets is the development of a coherent framework, which provides a new paradigm for analyzing data. We are now at the stage where it is possible to formulate this framework in an organized manner which is accessible to the broader community of scientists, engineers, and researchers, and not just wavelet experts.

One aspect of our research is to develop this framework, fueling the transition of wavelets from a specialized technique to a broadly used methodology. Our efforts in this direction have resulted in the writing of an article for publication in the IEEE Spectrum [BDG95]: see section 3.

Another aspect of research related to developing the "wavelets approach" is the design of S+WAVELETS toolkit. The design of the toolkit is closely linked to our efforts to develop a coherent framework for wavelet analysis. The design of the toolkit is discussed in the technical report [BG95b]. The toolkit design and the corresponding technical report were partially supported by this contract.

We will continue our research in this area, both in regards to writing new papers and developing software promoting the use and application of wavelets.

2.4.1 S+WAVELETS Toolkit

S+WAVELETS is an object-oriented toolkit for wavelet analysis of time series and images [BG94]. It is the only commercial toolkit for wavelet analysis incorporated into a high-level interactive language such as S-PLUS. A high-level overview of the toolkit is given in figures 14 and 15.

The primary objective of the S+WAVELETS toolkit is to provide a mature commercial wavelet analysis product to the engineering and scientific research community. The commercial software for S+WAVELETS is being developed under the support of NASA Phase II SBIR Contract No. NAS13-587. The S+WAVELETS toolkit provides important support for our research program with ONR:

- It provides an environment in which we can rapidly prototype and test new wavelet based methods which we develop in our ONR research. This includes not only testing methods to determine performance on artificial or real test data sets, but also evaluating the practical implementations on real data sets of interest to the Navy.

- The existence of our NASA SBIR funded S+WAVELETS commercial software development work means that we can provide China Lake (Gary Hewer and colleagues) with specialized software for wavelet analysis.

We will continue to develop software based on the toolkit for use in our research and by Navy researchers at China Lake. In particular, we will develop software for efficient wavelet processing of signal and image arrays and non-decimated 2-D wavelet transforms.

2.5 DIMENSION REDUCTION

Since wavelets are very effective at compacting energy, wavelets are good tools for *dimension reduction* in problems involving very large data sets. The dimension reduction property of wavelets is important for applications such as image compression, factor analysis, and numerical analysis: see chapter 11 of [Wic94].

We are investigating the use of wavelets as a dimension reduction tool for the problem of classification of transient acoustic signals (see below). In future research,

we may also look at use of dimension reduction as a means to apply other statistical techniques to very large data sets.

2.5.1 Algorithms for Classification

Following the work by Saito and Coifman [SC94], we illustrate the basic ideas of using wavelets to reduce the dimension of the problem of classifying transient acoustic signals. We excerpt three classes of signals: whale clicks, snapping shrimp and background noise. Figure 12 displays three samples of length 2048 from the three classes. The wavelet classification scheme is based on reducing the signals of dimension 2048 to vectors of length 6, and building a classification tree based on the length 6 vectors. Figure 13 shows the resulting classification tree.

Specifically, this tree was built as follows:

[1] 25 samples from each class are drawn to form a training data set.

[2] For each sample, a wavelet packet table is computed and then an energy map is obtained by squaring the wavelet packet table and rescaling by the total energy of the signal.

[3] The energy maps from each class are then combined to form a single energy map for the class.

[4] Distances between the energy maps (totally 3 in this example) are computed and combined to a single distance measure. An optimal basis which maximizes the distance is derived. The optimal basis in this example consists of crystals: *w4.0, w4.1, w4.2, w4.3, w2.1, w1.1.* L_2 distance is used in this example, other discrimination measures include *relative entropy* and general L_p *distance*, [SC94].

[5] The training samples are transformed to the optimal basis and total energy for each crystal is accumulated.

[6] A classification tree is grown based on the accumulated energy vectors (of length 6 in this example) and is displayed in Figure 13.

250 samples are drawn from each class and classified by the classification tree (apply step [5] above). The misclassification rate is 13.33%.

In future research, we will investigate some of the following improvements

[1] Shrink the energy maps to eliminate ambient noise and enhance features.

[2] Time-invariant discriminant measures, such as rank related (e.g. Kruskal-Wallis [BD77]) statistics for comparing the component-wise distributions of

the transform coefficients, should be used for time-invariant signals (like whale clicks). Since these discrimination measures may not be additive, new optimal basis selection algorithms need to be developed.

[3] For a given optimal transform, an equally challenge issue is how to compress the transformed coefficients. For time-variant signals, taking certain number of top coefficients, as suggested by Saito and Coifman, is a straight forward solution. However, when signals are time-invariant, this is not be the case: a transient may occur at any time and therefore the coefficients associated with the transient may occur at any time. In above example, we have used the accumulated energy. There are other options we can explore, such as top frequencies, autocorrelations and moments.

2.6 ALGORITHMS FOR WAVELET ANALYSIS

To support our research for analysis of Navy data, we have spent some effort in development of fundamental algorithms for wavelet analysis. This work has involved two separate projects:

- Investigation in the fundamental properties of certain biorthogonal wavelet functions.
- Development of algorithms to handle signals and images of arbitrary dimensions with a variety of boundary conditions.

This work is described in more detail below.

2.6.1 Fundamental Properties of Biorthogonal Wavelets

In development of the S+WAVELETS toolkit, we discovered a problem in graphing certain biorthogonal wavelets; namely, the wavelets in Tables 8.2 and 8.3 of the book by Daubechies [Dau92] with $(\tilde{N}, N) = (2,2)$, $(3,1)$, $(3,3)$, and $(5,5)$. In the toolkit, these are referred to as wavelets `bs2.2`, `bs3.1`, `bs3.3`, and `vs3`. It turns out that these wavelets are *infinite* at all dyadics! This fact may have important implications for analysis using these wavelets, since it implies a certain underlying instability.

We were naturally led to the question: if the wavelets are infinite at all dyadics, how do we graph the wavelets to get an understanding of their shape? The usual constructions for evaluating wavelet functions, described by Strang [Str89], do not apply.

In work partially supported by this contract, we developed an approach for graphing such wavelets. This approach formulates an eigenvalue equation based on the dilation equation evaluated at non-dyadic points. This work is described in a paper published in "Recent Advances in Approximation Theory, Wavelets and Applications" [RBG94].

2.6.2 Algorithms for Finite Signals

A very important problem in wavelet analysis of signals and images is the ability to handle arbitrary signal/image samples sizes with a variety of boundary conditions. In on-going research, we are developing a suite of algorithms to address this problem.

Most published research in wavelet analysis is based on signals and images which are of length/dimension 2^J. Unfortunately, particularly for images, we do not always have the luxury to restrict our sample size in that manner. Also, in wavelet analysis, the treatment of the boundaries is very important. For computational simplicity, it is often assumed the image or signal is periodic. This can lead, however, to very serious artifacts which cause problems in both statistical and data compression applications.

In a technical report, partially supported by this contract, we describe a suite of algorithms for addressing these issues [BG95a]. In work in progress, we are extending this work, further improving these algorithms and developing a conceptual framework for understanding finite wavelet operators [BGR95]: see section 3.

3 PAPERS AND TALKS

3.1 Published Papers

1. **Smoothing and Robust Wavelet Analysis.** Andrew G. Bruce, David L. Donoho, Hong-Ye Gao, and R. Douglas Martin. In *Proceedings in Computational Statistics* (invited paper), Vienna, Austria, August, 1994.

   In a series of papers, Donoho and Johnstone develop a powerful theory based on wavelets for extracting non-smooth signals from noisy data. Several nonlinear smoothing algorithms are presented which provide high performance for removing Gaussian noise from a wide range of spatially inhomogeneous signals. However, like other methods based on the linear wavelet transform, these algorithms are very sensitive to certain types of non-Gaussian noise, such as outliers. In this paper, we develop *outlier resistant* wavelet transforms. In these transforms, outliers and outlier patches are localized to just a few scales. By using the outlier resistant wavelet transforms, we improve upon the Donoho and Johnstone nonlinear signal extraction methods. The outlier resistant wavelet algorithms are included with the S+WAVELETS object-oriented toolkit for wavelet analysis.

2. **Nonlinear and Robust Wavelet Analysis.** Andrew Bruce, David L. Donoho, Hong-Ye Gao, and R. Douglas Martin. In *SPIE Proceedings, Wavelet Applications*, Orlando, Florida, April, 1994.

   See above for abstract.

3. **Non-smooth Wavelets: Graphing Functions Unbounded on Every Interval.** David Ragozin, Andrew Bruce, and Hong-Ye Gao. In *Recent Advances in Approximation Theory, Wavelets and Applications*, Kluwer, 1994.

   Several wavelets from well known biorthogonal families are shown to be unbounded on every interval. One, in fact, is shown to be infinite at each dyadic rational. Not withstanding these facts, we show how to compute exact values for these wavelets at many points and thus achieve exact pictures for these functions.

3.2 Papers Accepted for Publication

1. **Ideal Denoising in an Orthonormal Basis Selected from a Library of Bases** David L. Donoho and Iain M. Johnstone. To appear in *Comptes Rendus de l'Academie de Science Paris.*

   Suppose we have observations $y_i = s_i + z_i$, $i = 1, \ldots, n$, where (s_i) is signal and (z_i) is i.i.d. Gaussian white noise. Suppose we have avilable a library $\mathcal{L}$ of orthogonal bases, such as the Wavelet Packet bases or the Cosine Packet bases of Coifman and Meyer. We wish to select, adpatively based on the noisy data (y_i), a basis in which best to recover the signal ("de-noising"). Let M_n be the total number of distinct vectors occurring among all bases in the library and let $t_n = \sqrt{2\log(M_n)}$. (For wavelet packets, $M_n = n\log_2(n)$.)

   Let $y[\mathcal{B}]$ denote the original data y transformed into the Basis $\mathcal{B}$. Choose $\lambda > 8$ and set $Lambda_n = (\lambda \cdot (1 + t_n))^2$. Define the entropy functional

   $$\mathcal{E}_\lambda(y, \mathcal{B}) = \sum_i \min(y_i^2[\mathcal{B}], \Lambda_n^2).$$

   Let $\hat{\mathcal{B}}$ be the best orthogonal basis according to this entropy:

   $$\hat{\mathcal{B}} = \arg\min_{\mathcal{B}\in\mathcal{L}} \mathcal{E}_\lambda(y, \mathcal{B}).$$

   Define the hard-threshold nonlinearity $\eta_t(y) = y 1_{\{|y|>t\}}$. In the empirical best basis, apply hard-thresholding with threshold $t = \sqrt{\Lambda_n}$:

   $$\hat{s}_i^*[\hat{\mathcal{B}}] = \eta_{\sqrt{\Lambda_n}}(y_i[\hat{\mathcal{B}}]).$$

   Theorem: With probability exceeding $\pi_n = 1 - e/M_n$,

   $$\|\hat{s}^* - s\|_2^2 \leq (1 - 8/\lambda)^{-1} \cdot \Lambda_n \cdot \min_{\mathcal{B}\in\mathcal{L}} E\|\hat{s}_{\mathcal{B}} - s\|_2^2.$$

   Here the minimum is over all ideal procedures working in all vases of the library, i.e. in basis $\mathcal{B}$, $\hat{s}_{\mathcal{B}}$ is just $y_i[\mathcal{B}]1_{\{|s_i[\mathcal{B}]>1\}}$.

In short, the basis-adaptive estimator achieves a loss within a logarithmic factor of all the ideal risk which would be achievable if one had available an oracle which would supply perfect information about the ideal basis in which to de-noise, and also about which coordinates were large or small.

The result extends in obvious ways to more general orthogonal basis libraries, basically to any libraries constructed from an at-most polynomially-growing number of coefficient functionals. Parallel results can be developed for closely related entropies.

2. **On Minimum Entropy Segmentation.** David L. Donoho. To appear in *Wavelet Analysis and Its Applications*, edited by Charles Chui.

   A segmented multiresolution analyses of $[0,1]$ is described. Such multiresolution analyses lead to segmented wavelet bases which are adapted to discontinuities, cusps, etc., at a given location $\tau \in [0,1]$. The approach emphasizes the idea of *average-interpolation* – synthesizing a smooth function on the line having prescribed boxcar averages. This particular approach leads to methods with *subpixel resolution* and to wavelet transforms with the advantage that, for a signal of length n, all n pixel-level segmented wavelet transforms can be computed simultaneously in a total time and space which are both $O(n\log(n))$.

   The search for a segmented wavelet basis is considered which, among all such segmented bases, minimizes the "entropy" of the resulting coefficients. Fast access to all segmentations enables fast search for a best segmentation.

   When the "entropy" is Stein's Unbiased Risk Estimate, one obtains a new method of edge-preserving de-noising. When the "entropy" is the ℓ^2-energy, one obtains a new multi-resolution edge detector, which works not only for step discontinuities but also for cusp and higher-order discontinuities, and in a near-optimal fashion in the presence of noise.

   An iterative approach is also described, *Segmentation Pursuit*, for identifying edges by the fast segmentation algorithm and removing them from the data.

3.3 Papers Accepted Pending Review

1. **Wavelet Analysis Tools Become Widely Available.** Andrew G. Bruce, David L. Donoho, and Hong-Ye Gao. Under review at the IEEE Spectrum.

   This is an introductory article intended to promote the use and application of wavelets and wavelet software. The article gives a non-mathematical review of wavelet analysis, complete with color graphics and informative sidebars. Several applications of wavelets are given, including data compression, noise removal, and development of fast algorithms. The discussion includes the

latest developments in wavelets, including time-frequency decompositions such as wavelet packets and cosine packets.

The main focus of the article is to give a description of the "wavelets toolkit", which is the collection of software tools needed to perform a wavelet analysis. A wide range of commercial and public domain software is reviewed. The advantages and disadvantages of different wavelet computing environments are discussed.

2. **Choice of Thresholds for Wavelet Shrinkage Estimate of the Spectrum.** Hong-Ye Gao. Under review at the Journal of Time Series Analysis.

   We study the problem of estimating the log spectrum of a stationary Gaussian time series by thresholding the empirical wavelet coefficients. We propose the use of thresholds $t_{j,n}$ depending on sample size n, wavelet basis ψ and resolution level j. At fine resolution levels ($j = 1, 2, ...$), we propose

$$t_{j,n} = \alpha_j \log n,$$

   where $\{\alpha_j\}$ are level-dependent constants and at coarse levels ($j \gg 1$)

$$t_{j,n} = \frac{\pi}{\sqrt{3}} \sqrt{\log n}.$$

   The purpose of this thresholding level is to make the reconstructed log-spectrum as nearly noise-free as possible. In addition to being pleasant from a visual point of view, the noise-free character leads to attractive theoretical properties over a wide range of smoothness assumptions. Previous proposals set much smaller thresholds and did not enjoy these properties.

3.4 Technical Reports

1. **Wavelet and Isotonic Regression.** Hong-Ye Gao.

   Consider the model:

$$y_i = f(t_i) + z_i$$

   where f is a decreasing function and $\{z_i\}$ are assumed to be a stationary Gaussian process with mean zero and variance σ^2. We propose a simple thresholding procedure based on the fact that the wavelet coefficients for f, under Haar basis, are non-negative. We show that our estimator is competitive with the Grenander estimator both theoretically and numerically (in the sense of mean-square-error).

   To go beyond Haar wavelets, new spline wavelet refinement operators are developed. These operators preserve the monotonicity of the refinement.

2. **S+WAVELETS: Algorithms and Technical Details.** Andrew Bruce, Hong-Ye Gao, and David Ragozin.

   A complete description is given for the algorithms in S+WAVELETS software toolkit for wavelet analysis. These algorithms include wavelet transforms, wavelet packet transforms, cosine packet transforms, and non-decimated wavelet transforms. Implementations for the transforms and their inverses are given for a variety of boundary treatment rules, including periodic, reflection, interval wavelets (Cohen et al. [CDV93]), and zero/polynomial extension. In addition, modifications to the standard algorithms are given to handle signals or images with dimensions not divisible by a power of two.

3.5 Talks and Presentations

1. **SPIE Meetings.** *Denoising and Robust Non-Linear Wavelet Analysis*, April, 1994, Orlando, FL.

2. **CompStat Meeting**, *Smoothing and Robust Wavelet Analysis*, August, 1994, Vienna, Austria.

3. **IMS Talks.** *An Object-Oriented Toolkit for Wavelet Analysis*, April, 1994, Cleveland, OH.

3.6 Papers In Preperation

1. **Wavelet Packet Transforms for Finite Signals.** Andrew Bruce, Hong-Ye Gao, and David Ragozin.

   As originally developed by [Dau88] the discrete wavelet transform is defined on the entire real line. To apply the wavelet and wavelet packet transforms to a finite signal, several approaches have been pursued:

   (a) An *ad hoc* rule is used for applying the wavelet filters at the boundaries of the signal, typically by recursively extending the signal at each iteration in the wavelet decomposition.

   (b) The signal is extended infinitely according to some boundary rule (e.g., periodic extension) and sufficient coefficients are retained to reproduce the transform (see, for example, [MT92, BF92]).

   (c) A new wavelet transform is formally defined on compact sets (e.g., [CDV93] define an orthogonal wavelet basis on compact intervals).

   A unified framework is developed for these approaches. The methods are compared, both theoretically and empirically with emphasis on their implications

for data analysis and statistical estimation. Practical algorithms are given for software implementation.

2. **A Fast, Robust, Nonlinear Wavelet Transform.** Andrew Bruce and David L. Donoho.

   A triadic nonlinear refinement scheme is described based on solving the following problem: given the median value of a function on blocks of size 3^{-j}, impute medians to all finer-scale blocks of size 3^{-j-1} by fitting local polynomials to the neighboring block medians and calculating the block medians of the polynomials one scale finer. This nonlinear refinement scheme is practical for linear and quadratic polynomials. It is a nonlinear cousin of Deslauriers-Dubuc interpolation and of Average-Interpolation. It is reproduces polynomials up to the degree of the fit.

   This refinement scheme leads to nonlinear multiresolution analyses. The wavelet coefficient functionals are possibly nonlinear combinations of a finitely many block medians at the same scale; a fast nonlinear algorithm to compute all block medians in $O(nlog(n))$ time can be made by a simple merging of sorted lists. Using this, one gets a fast nonlinear wavelet transform.

   An advantage of the transform is its robustness against outliers and impulsive noise, while preserving high degree approximation quality.

3. **Spectral Density Estimation by Wavelet Shrinkage.** Hong-Ye Gao.

   We study the problem of estimating the spectral density of a stationary Gaussian time series. We use an orthogonal wavelet system whose members are periodic functions and have a finite number of non-zero Fourier coefficients – periodized Meyer wavelets. We apply shrinkage rules to the empirical wavelet coefficients. We show that estimates based on thresholds $t_{j,n} = \lambda_j \log n$ for certain λ_j, with n the sample size, have near-optimal L_2 convergence rates, over any Besov class in a wide range. In some cases, which includes the Bump Algebra, wavelet shrinkage procedures significantly outperform classical linear procedures, such as window methods and AR approximation methods.

4. **S+WAVELETS: Applications I – Wavelet De-Noising.** Andrew Bruce and Hong-Ye Gao.

   This is the first in a series of papers illustrating applications of wavelets using S+WAVELETS. This paper focuses on the removal of noise in a variety of applications. Some of the problems examined include noise removal from a noisy NMR signal, signal extraction from radar glint noise, variance estimate for wavelet de-noising, image enhancement, probability and spectral density estimation, enhancement of noisy speech data, and isotonic regression.

5. **S+WAVELETS: Applications II – Data Compression.** Andrew Bruce and Hong-Ye Gao.

   This is the second in a series of papers illustrating applications of wavelets using S+WAVELETS. Several applications of wavelets to problems in data compression are presented. Some of the problems include compression of seismic signals, acoustic signals, digital MR images, and digital fingerprint images.

6. **S+WAVELETS: Applications III – Signal Processing.** Andrew Bruce and Hong-Ye Gao.

   This is the third in a series of papers illustrating applications of wavelets using S+WAVELETS. A variety of signal processing problems is examined, including time-frequency analysis of acoustic signals, singularity detection and estimation, and fast algorithms for signal classification.

4 FUTURE RESEARCH PLANS

Our future research plans will continue the six research areas as discussed in section 2. We will also complete the papers in preparation discussed in section 3. In addition, as time permits, we will pursue some new ideas in these areas, including

Fast L_1/M-estimators for wavelets: In the first year of our research, we found that using more robust methods for fitting wavelets produced valuable decompositions for many types of signals and images. However, these methods were of limited value since they required extensive computational effort. Using well known stochastic approximation formula, we propose to develop approximate methods for fast computation of L_1/M-estimators for wavelet decompositions.

Robust time series filters. The ARMA interpretation of the wavelet filters will be exploited to yield a robust wavelet decomposition using the approximate condition mean (ACM) smoothers developed by C. J. Masreliez [Mas75] and R. D. Martin [Mar79, MF93].

Robust wavelets in higher dimensions. The robust algorithms developed for one-dimensional signals will be extended to higher-dimensional data, following the work of [DJL92, Ric93]. One and two dimensional non-decimated algorithms for robust wavelet analysis will also be explored.

Wavelet coefficients for fractional Brownian motion. There has been considerable activity examining the statistical properties and utility of wavelet coefficients for fractional Brownian motion. In particular, there is work on texture analysis [TD90], estimation and analysis of fractional Brownian motion [DT91, Fla92],

extracting fractal signals from noisy measurements [WO92], correlation structure of wavelet coefficients for fractional Brownian motion [TK92, DM94], Karhunen-Loeve like expansions for $1/f$ processes [Wor92], and analysis of long-memory processes [PG94]. These results give some promise for further statistical results.

Simulation, bootstrapping, and modeling of non-Gaussian processes. Simulation of non-Gaussian processes has many practical uses in Navy applications. The use of wavelets gives hope of providing a convenient method for simulating non-Gaussian random processes. One would compute the wavelet coefficients for an observed sample of a non-Gaussian process, and then generate simulation sample paths by appropriate random sampling of the wavelet coefficients and subsequent expansion in terms of wavelets. Various methods of random sampling need to be studied. Then one needs to establish that the sample paths generated converge in a suitable probabilistic sense to the probability measure for the observed sample path used to calculate the wavelet coefficients as the sample size tends to infinity. Work in this area could lead to an effective *bootstrap* method for carrying out inference for non-Gaussian processes. This will be applied to improve the variance estimation methods for WaveShrink estimators.

5 FIGURES

List of Figures

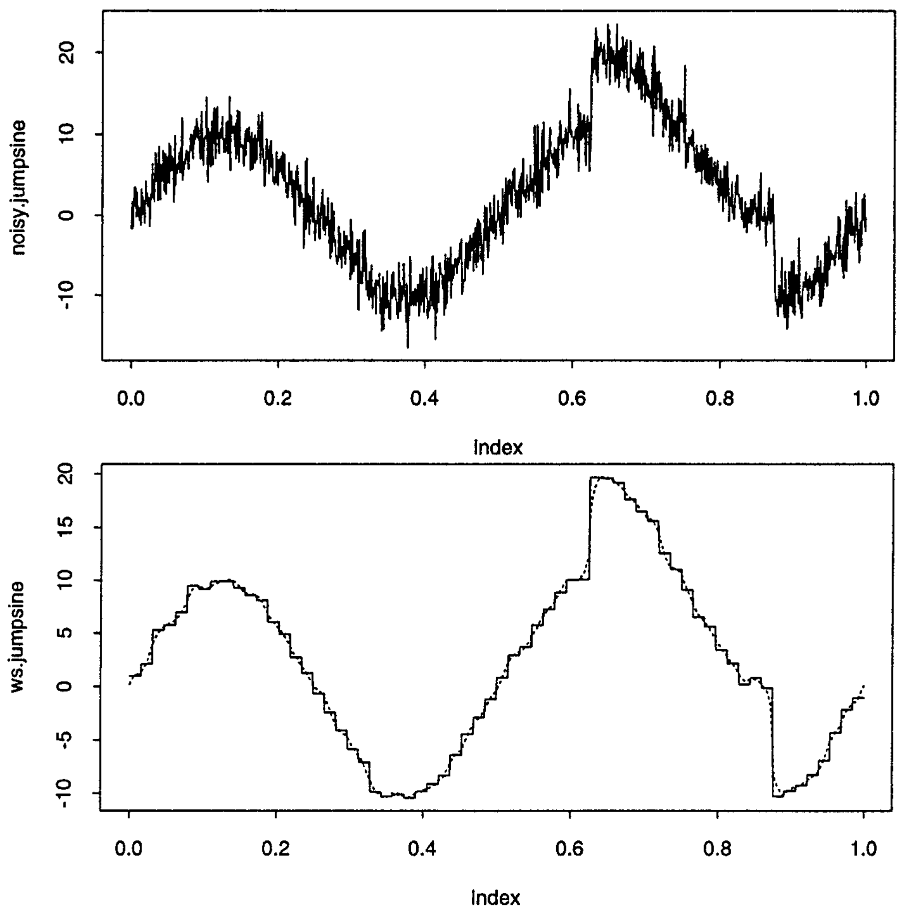

Figure 1: WaveShrink with non-decimated wavelets. Top: the noisy jumpsine signal. Bottom (solid line): smooth produced by wavelet shrinkage using the orthogonal DWT. Bottom (dashed line): smooth produced by wavelet shrinkage using the non-decimated DWT. The `haar` wavelet is used in this example.

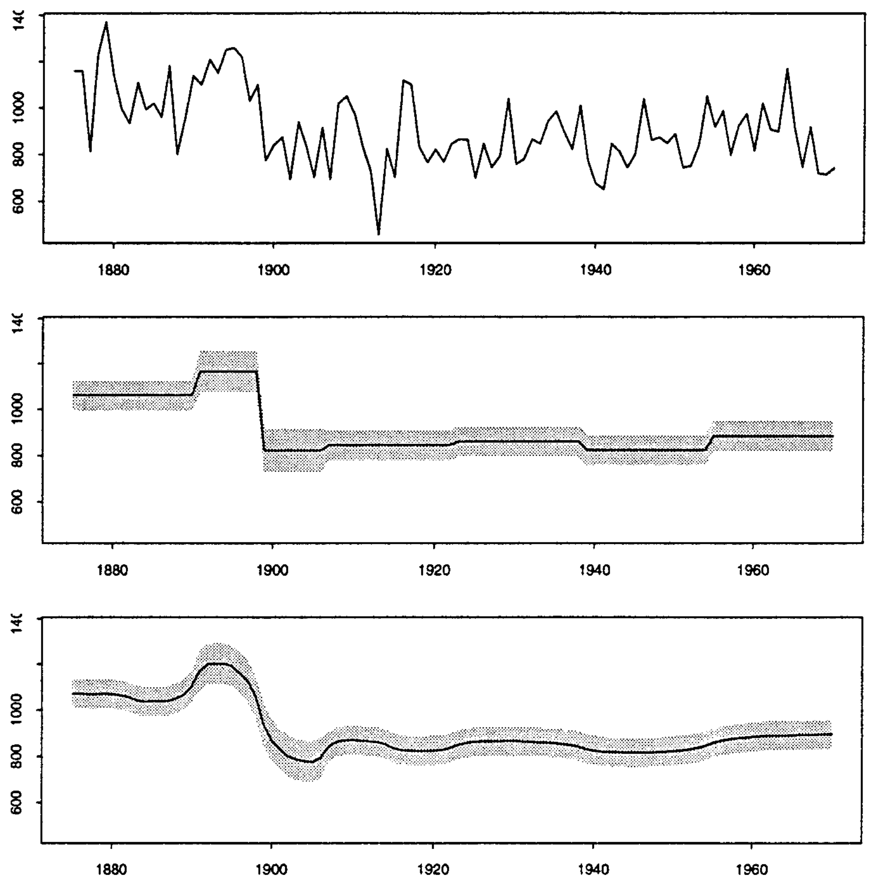

Figure 2: Approximate confidence band for WaveShrink. Top: the annual Nile River flow at Aswan from 1875 to 1970. Middle: WaveShrink using the Haar wavelet with an approximate 95% confidence band. Bottom: WaveShrink using the `bs1.3` wavelet with an approximate 95% confidence band. The confidence interval clearly shows a significant drop around the time of the building of the Aswan dam in 1899-1902.

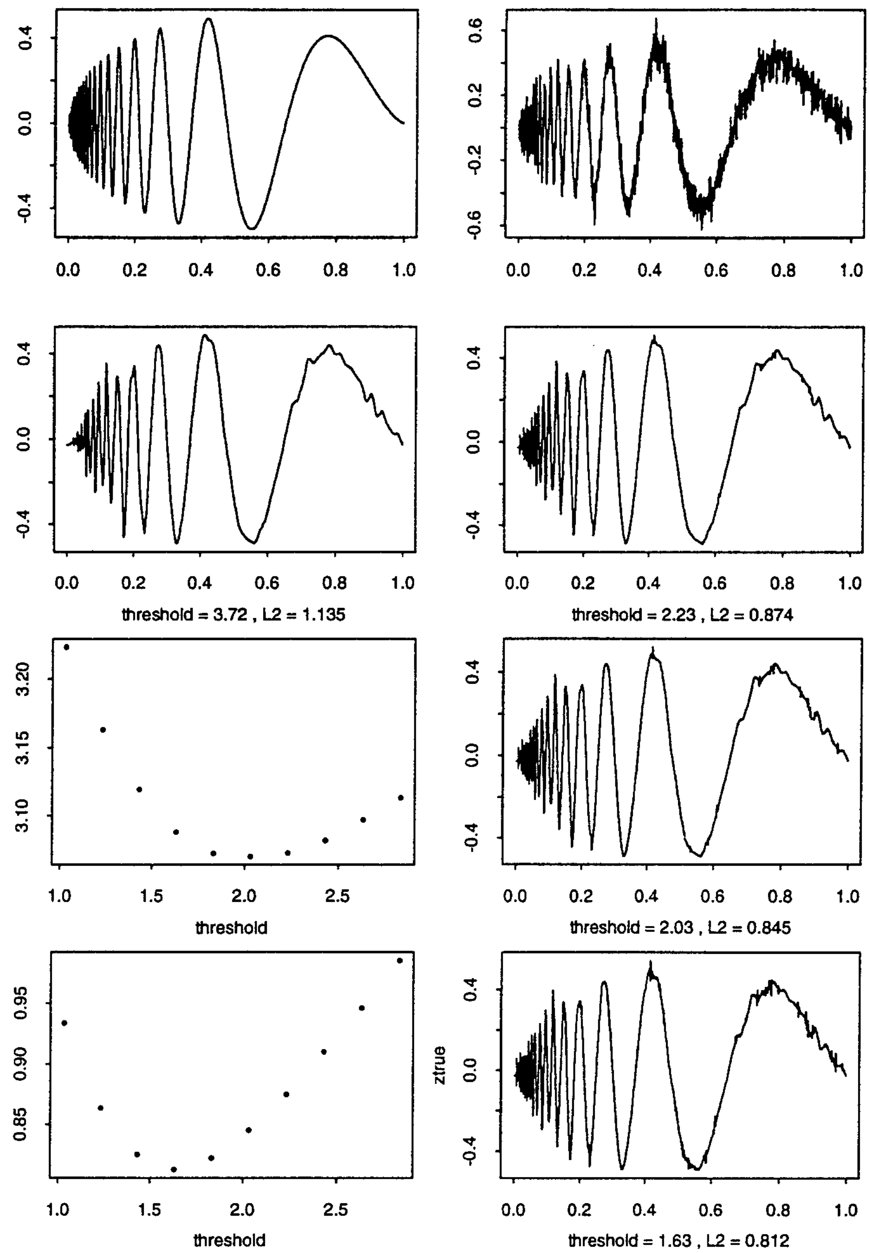

Figure 3: First row: the doppler signal (left) and the noisy doppler signal (right). Second row: the WaveShrink estimates using the "universal" threshold (left) and the "minimax" threshold (right). Third row: the mean-square error (MSE) of the WaveShrink estimate of *every other* sample value based on the remaining half of the data for a range of thresholds from 1 to 3 (left) and the cross-validation WaveShrink estimate which achieves the minimum MSE (right). Fourth row: MSE between the WaveShrink estimate and the true doppler signal using thresholds ranging from 1 to 3 (left) and the WaveShrink estimate using the threshold which achieves the minimum (right). Cross validation consistently leads to lower MSE than the "universal" and "minimax" thresholds for this example.

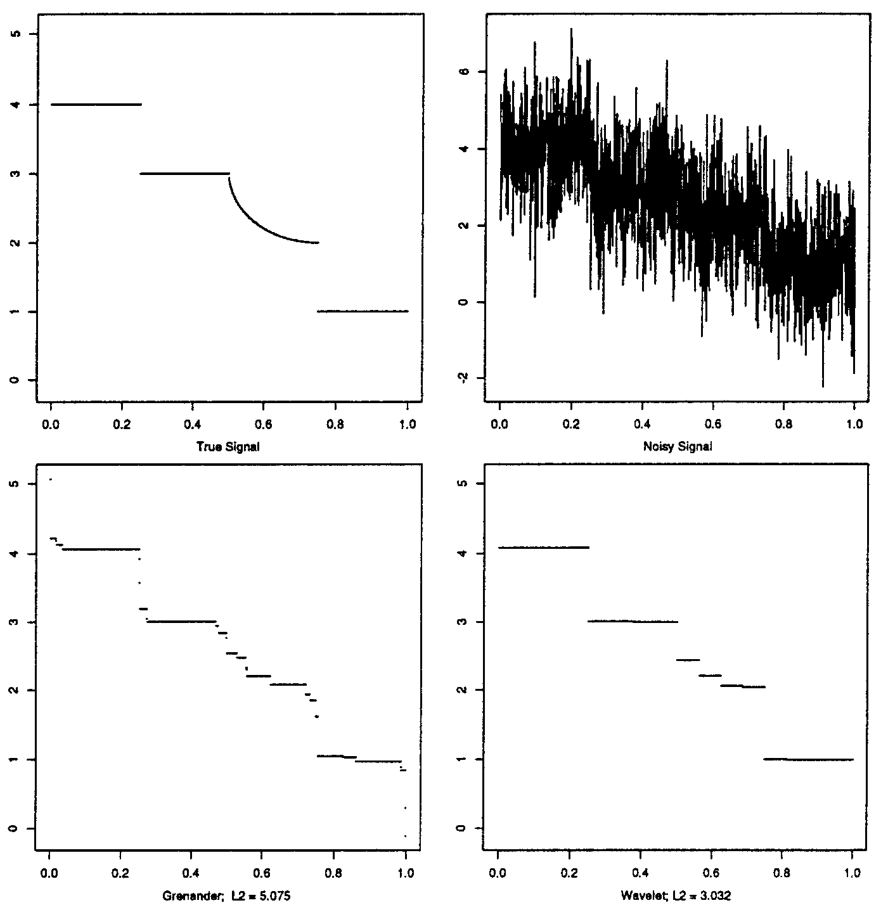

Figure 4: Comparison of wavelet estimate and Grenander estimate for monotone signal. Top left: a decreasing curve with two discontinuities, Top right: the same curve plus Gaussian white noise, Bottom left: the Grenander estimate with mean-square error, Bottom right: the wavelet estimate with mean-square error. The wavelet estimate has lower mean square error and does a better job of preserving the jumps.

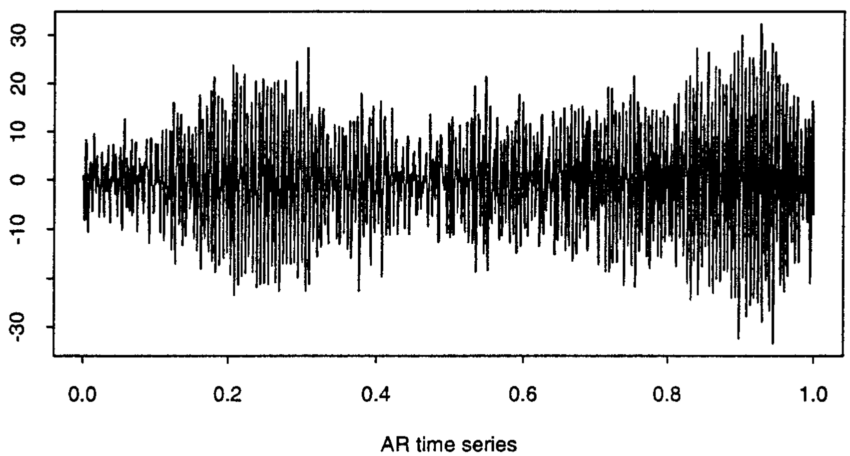

Figure 5: A simulated AR series which has sharp peaks in its spectral density function.

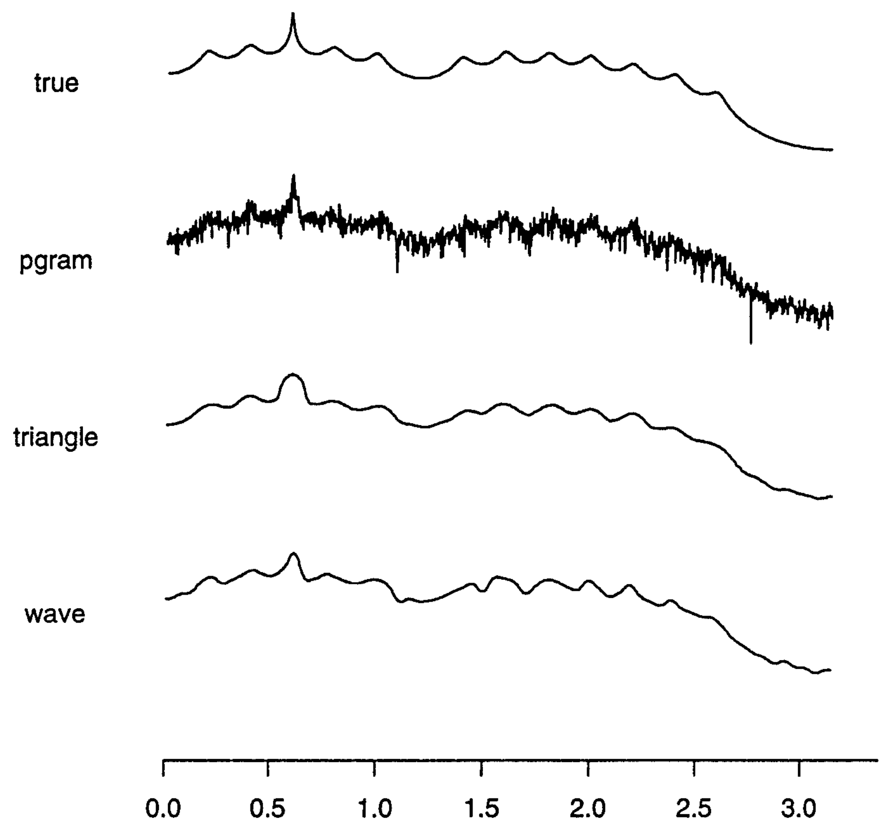

Figure 6: The true spectrum for the simulated AR series of figure 5 (top), the raw log-periodogram (second), the smoothed log-periodogram using a triangular spectral window (third), and the WaveShrink estimate (bottom).

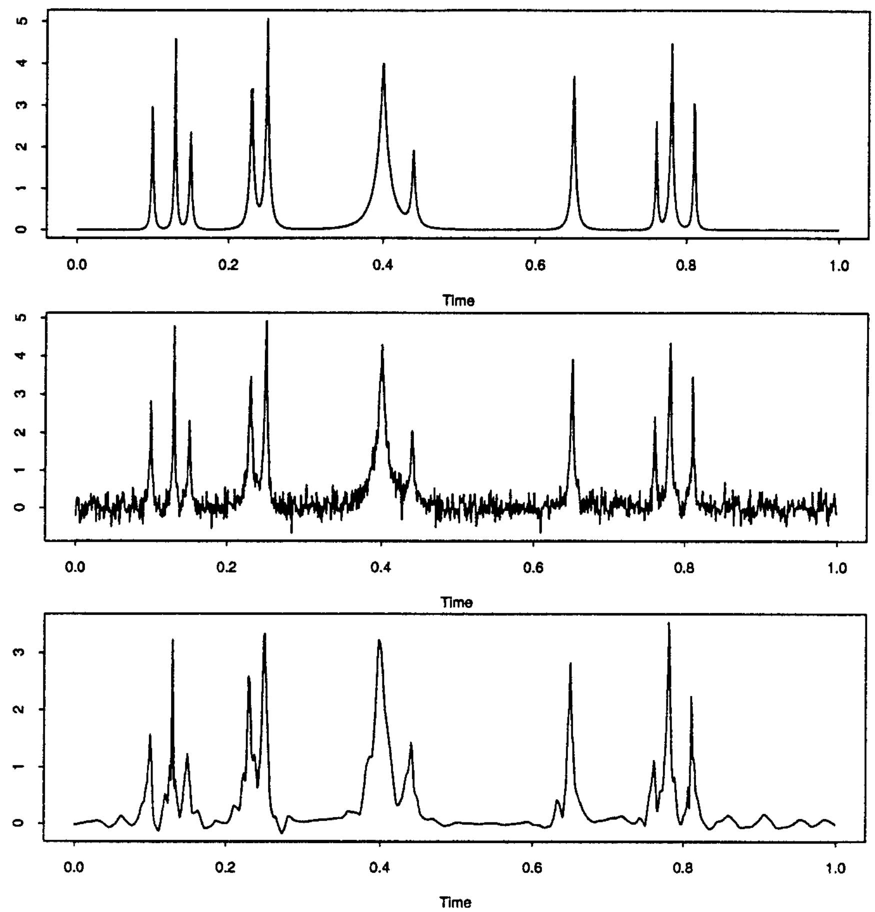

Figure 7: The bumps signal (top), the bumps signal plus Gaussian white noise (middle), and an estimated of the bumps signal obtained from WaveShrink (bottom).

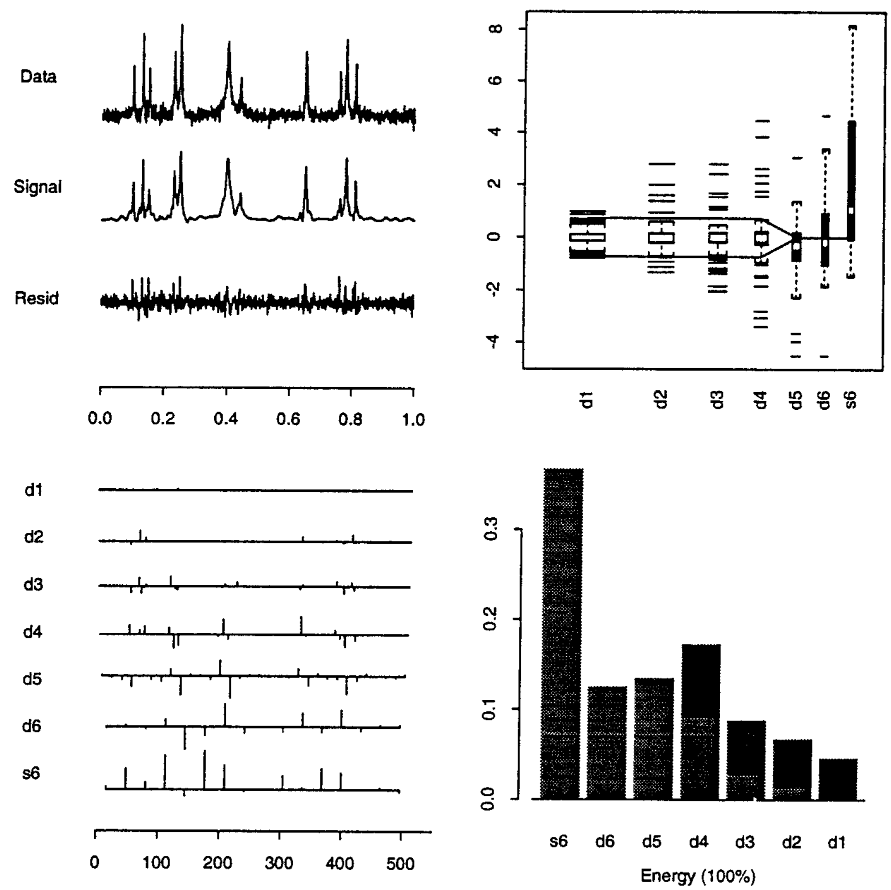

Figure 8: Visualizing the WaveShrink fit for the bumps signal. Top left: the decomposition of the data into signal plus noise. Top right: Boxplots of the DWT coefficients for the original data with the WaveShrink thresholds superimposed as horizontal lines. Any wavelet coefficient lying inside the lines is set to zero. Bottom left: The DWT of the signal. Bottom right: A barplot showing the decomposition of the energy of the data into the energy attributable to signal and residual energy.

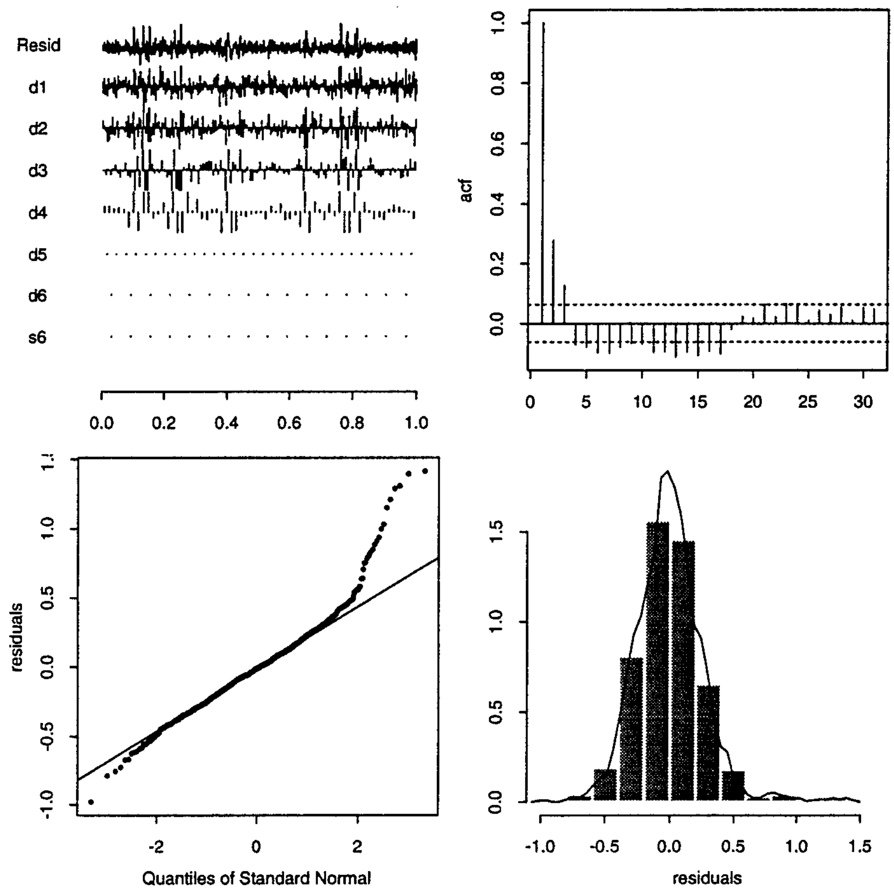

Figure 9: Four views of the residuals from the WaveShrink estimate of the bumps signal. Top left: the DWT of the residual component. Top right: the autocorrelation function (acf) of the residual component. Bottom left: the quantile-quantile plot of the residual component versus the quantiles of a standard normal. Bottom right: a histogram and density estimate of the residuals.

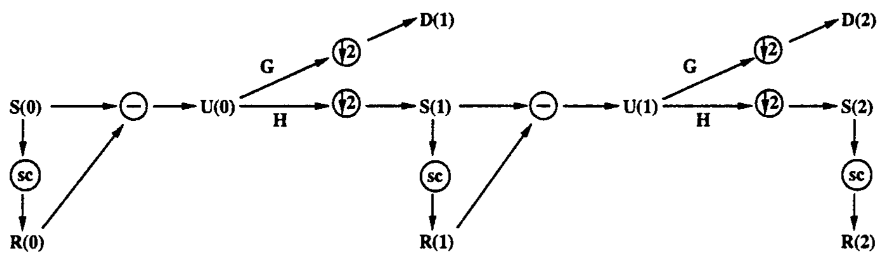

Figure 10: The robust smoother algorithm produces a pyramid decomposition with an extra component: the robust residual $R(\ell)$. For each multiresolution level, the low-pass coefficients $S(\ell)$ are first cleaned using a robust smoother cleaner, denoted by sc in the figure. The residuals are saved in the $R(\ell)$. The usual wavelet filters are then applied to the cleaned $S(\ell)$ to obtain $S(\ell+1)$ and $D(\ell+1)$.

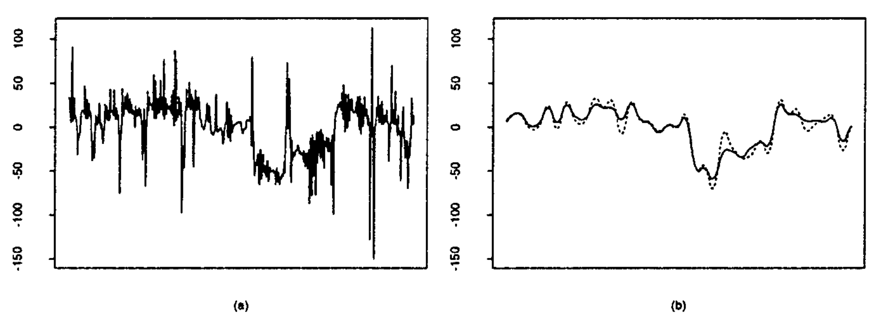

Figure 11: (a) Radar glint noise data in degrees, and (b) denoising by linear shrinkage of wavelets (dashed line) compared with denoising by WaveShrink combined with robust smoother-cleaner wavelets (solid line).

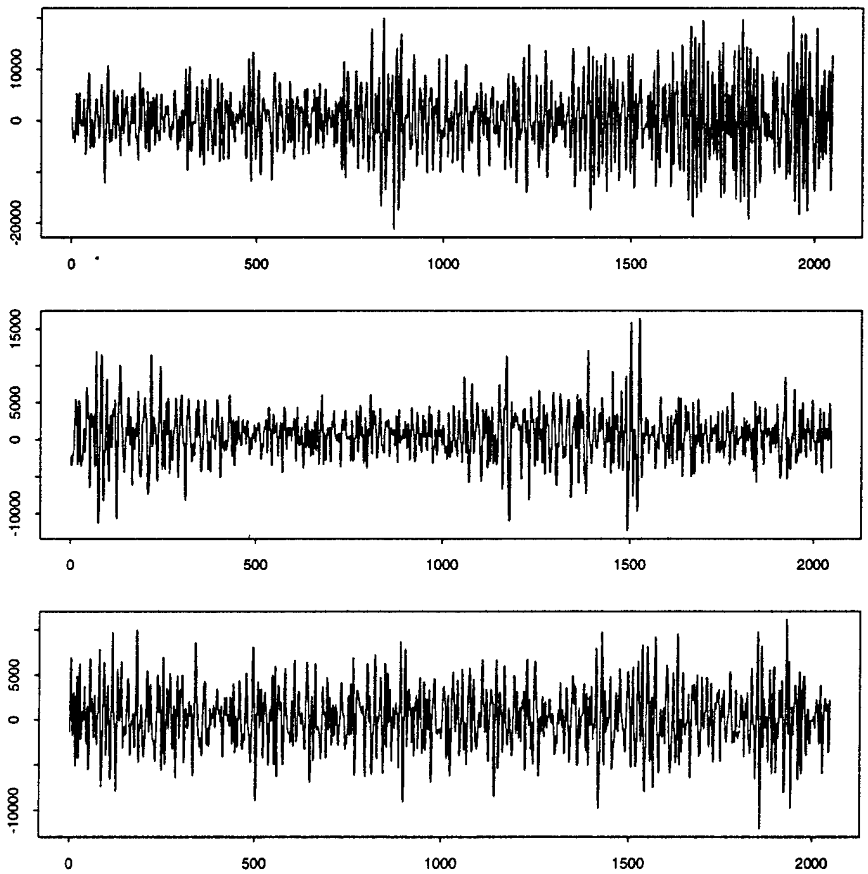

Figure 12: Three acoustic signals: whale clicks (top), shrimp clicks (middle), and background noise (bottom).

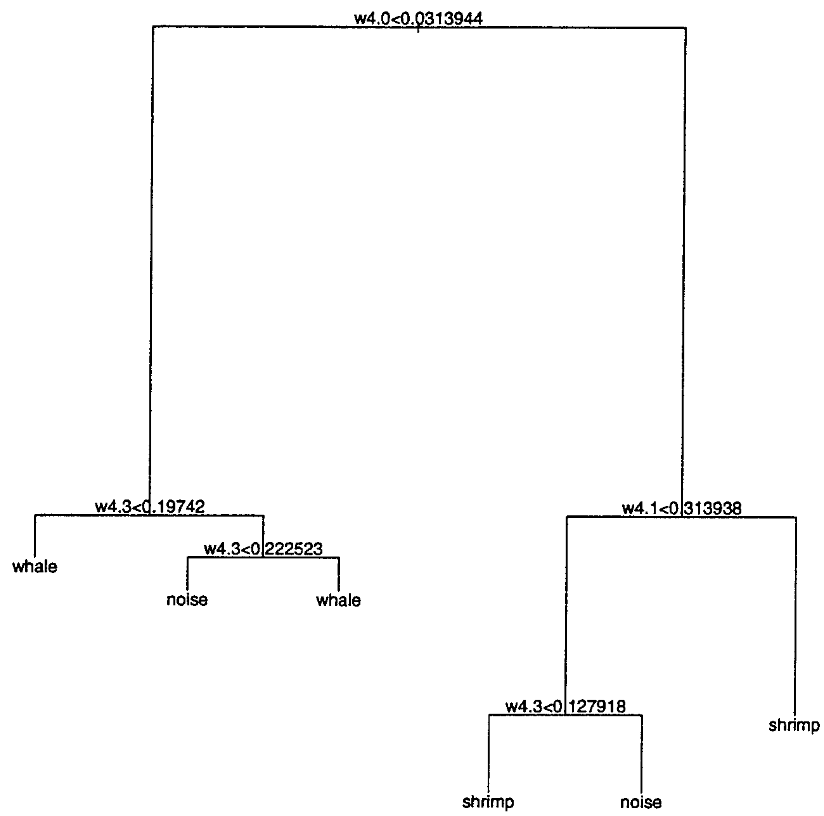

Figure 13: A classification tree for the acoustic signals derived from a wavelet packet energy map.

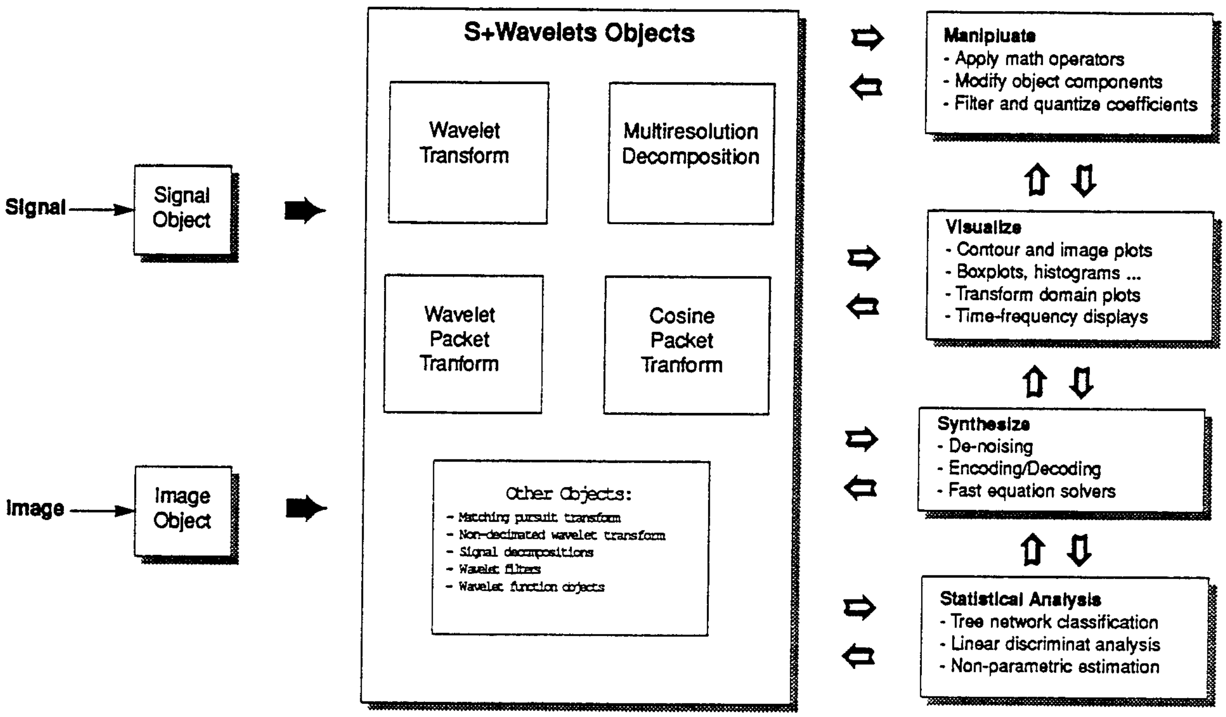

Figure 14: A wavelet analysis starts with the transformation of a signal or image object into a wavelet object. A wavelet representation is not a single transformation, but involves a wide range of objects. The toolkit provides a rich infrastructure for interacting with the data and wavelet objects. The user can visualize, manipulate and synthesize wavelet objects with built-in functions and operators. Because S+WAVELETS is embedded in S-PLUS, users can also apply a wide collection of statistical analysis tools, such as tree network classification, linear discriminant analysis, or non-linear additive models.

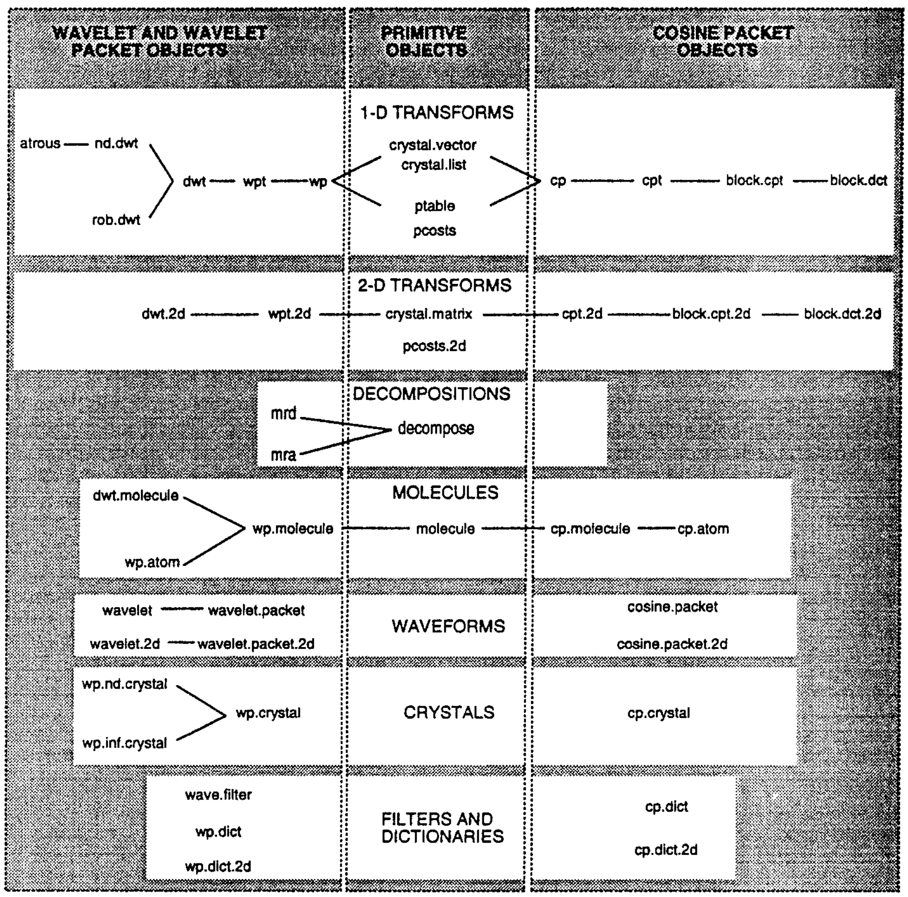

Figure 15: An overview of the object-oriented design of S+WAVELETS. The horizontal boxes group the objects by type: 1-D transforms, 2-D transforms, etc.. The objects are also categorized as primitive objects (middle vertical panel), objects specialized for wavelet and wavelet packet analysis (left vertical panel), and objects specialized for cosine packet analysis (right vertical panel). The lines indicate the inheritance hierarchy.

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